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SOME INEQUALITIES FOR MULTIVARIATE NORMAL DISTRIBUTION*

by

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Introduction.

Dunn [see 3] conjectured the following result: If $\underset{\sim}{X} \cap N_p(\underset{\sim}{0}, \Sigma)$, then

$$P[|X_i| \leq c_i,\ i = 1,\ldots, p] \geq \prod_{i=1}^{p} P[|X_i| \leq c_i].$$

This is later proved by Sidak [see 3] and Scott [3] and generalized by Khatri [2] as follows:

$$P[\underset{\sim}{X}^{(1)} \in D_1,\ \underset{\sim}{X}^{(2)} \in D_2] \geq P[\underset{\sim}{X}^{(1)} \in D_1]\ P[\underset{\sim}{X}^{(2)} \in D_2]$$

where $\underset{\sim}{X} = \binom{\underset{\sim}{X}^{(1)}}{\underset{\sim}{X}^{(2)}}$, provided $\mathrm{Cov}\ (\underset{\sim}{X}^{(1)}, \underset{\sim}{X}^{(2)})$ is of rank 1, and D_1, D_2 are convex sets in the spaces of $\underset{\sim}{X}^{(1)}$ and $\underset{\sim}{X}^{(2)}$, respectively, symmetric about the origin. Moreover, Khatri proved that

$$P(|X_i| \geq c_i,\ i = 1,\ldots, p) \geq \prod_{i=1}^{p} P[|X_i| \geq c_i]$$

if $\Sigma = D + \underset{\sim}{\alpha}\underset{\sim}{\alpha}'$ where D is diagonal and p.d. Scott [3] had already proved this result without the restriction on Σ.

It is shown in this note that Khatri's second result (without having the restriction on Σ) follows from his first result. It is further proved that Khatri's first result holds without having the restriction on $\mathrm{Cov}\ (\underset{\sim}{X}^{(1)}, \underset{\sim}{X}^{(2)})$. A few other simple results are also given. The main idea for all the proofs is derived from Scott's paper [3].

Results on inequalities for multivariate normal distribution.

Theorem 1:

Let $\underset{\sim}{X} = \binom{\underset{\sim}{X}^{(1)}}{\underset{\sim}{X}^{(2)}} \cap N_p(\underset{\sim}{0}, \Sigma)$. Then

$$P[\underset{\sim}{X}^{(1)} \notin D_1,\ \underset{\sim}{X}^{(2)} \notin D_2] \geq P[\underset{\sim}{X}^{(1)} \notin D_1]\ P[\underset{\sim}{X}^{(2)} \notin D_2]$$

provided $\mathrm{Cov}\ (\underset{\sim}{X}^{(1)}, \underset{\sim}{X}^{(2)})$ is of rank 1 and D_1, D_2 are convex sets, symmetric about the origin, in the spaces of $\underset{\sim}{X}^{(1)}$ and $\underset{\sim}{X}^{(2)}$, respectively.

Proof:

Consider a random vector $\underset{\sim}{Y}$ which has the same distribution as that of $\underset{\sim}{X}^{(1)}$ and independent of $\underset{\sim}{X}^{(2)}$.

$$P[\underset{\sim}{X}^{(1)} \in D_1] = P[\underset{\sim}{X}^{(1)} \in D_1, \underset{\sim}{X}^{(2)} \in D_2] + P[\underset{\sim}{X}^{(1)} \in D_1, \underset{\sim}{X}^{(2)} \notin D_2]$$

$$\geq P[\underset{\sim}{Y} \in D_1, \underset{\sim}{X}^{(2)} \in D_2] + P[\underset{\sim}{X}^{(1)} \in D_1, \underset{\sim}{X}^{(2)} \notin D_2]$$

from Khatri's first result.

$$P[\underset{\sim}{X}^{(1)} \in D_1] = P[\underset{\sim}{Y} \in D_1]$$

$$= P[\underset{\sim}{Y} \in D_1, \underset{\sim}{X}^{(2)} \in D_2] + P[\underset{\sim}{Y} \in D_1, \underset{\sim}{X}^{(2)} \notin D_2].$$

Combining the above two relations,

$$P[\underset{\sim}{X}^{(1)} \in D_1, \underset{\sim}{X}^{(2)} \notin D_2] \leq P[\underset{\sim}{Y} \in D_1, \underset{\sim}{X}^{(2)} \notin D_2].$$

Hence, subtracting from $P(\underset{\sim}{X}^{(2)} \notin D_2)$,

$$P[\underset{\sim}{X}^{(1)} \notin D_1, \underset{\sim}{X}^{(2)} \notin D_2] \geq P[Y \notin D_1, \underset{\sim}{X}^{(2)} \notin D_2]$$

$$= P[\underset{\sim}{X}^{(1)} \notin D_1]P[\underset{\sim}{X}^{(2)} \notin D_2].$$

Corollary 1.1:

Let $\underset{\sim}{X} \cap N_p(\underset{\sim}{0}, \Sigma)$. Then

$$P[|X_i| \geq c_i, \ i = 1, \ldots, p] \geq \prod_{i=1}^{p} P[|X_i| \geq c_i]$$

where $\underset{\sim}{X}' = (X_1, \ldots, X_p)$.

Proof:

This is obtained by repeated application of Theorem 1 by taking $\underset{\sim}{X}^{(2)}$ as one of components of $\underset{\sim}{X}$.

Theorem 2:

Let $\underset{\sim}{X} = \binom{\underset{\sim}{X}^{(1)}}{\underset{\sim}{X}^{(2)}} \cap N_p(\underset{\sim}{0}, \Sigma)$. Then

$$P[\underset{\sim}{X}^{(1)} \in D_1, \underset{\sim}{X}^{(2)} \in D_2] \geq P[\underset{\sim}{X}^{(1)} \in D_1]P[\underset{\sim}{X}^{(2)} \in D_2],$$

where D_1 and D_2 are two convex sets symmetric about the origin, in the spaces of $\underset{\sim}{X}^{(1)}$ and $\underset{\sim}{X}^{(2)}$ respectively.

Corollary 2.1:

Theorem 1 holds without having the restriction on Cov $(\underset{\sim}{X}^{(1)}, \underset{\sim}{X}^{(2)})$.

Theorem 2 will be proved using the following lemma [1].

Lemma 1:

Let $\underset{\sim}{U} \cap N_p(\underset{\sim}{0}, \Gamma_1)$, $\underset{\sim}{V} \cap N_p(\underset{\sim}{0}, \Gamma_2)$, and D be a convex and symmetric (about the origin) subset of the p-dimensional Euclidean space. Then

$$P[\underset{\sim}{V} \in D] \leq P[\underset{\sim}{U} \in D]$$

if $\Gamma_1 - \Gamma_2$ is positive semi-definite.

Proof of Theorem 2:

Let $\Sigma = TT'$, where

$$T = \left[\begin{array}{c|c} T_1 & T_2 \\ \hline 0 & T_3 \end{array}\right]$$

where the row-partition of T_1 is the same as the row-partition of $\underset{\sim}{X}$ and T_1 is square matrix. Define

$$\underset{\sim}{Y} = \left[\begin{matrix}\underset{\sim}{Y}^{(1)} \\ \underset{\sim}{Y}^{(2)}\end{matrix}\right] = T^{-1}\underset{\sim}{X}.$$

Then $\underset{\sim}{Y} \cap N_p(\underset{\sim}{0}, I_p)$ and $\underset{\sim}{X}^{(1)} = T_1\underset{\sim}{Y}^{(1)} + T_2\underset{\sim}{Y}^{(2)}$, $\underset{\sim}{X}^{(2)} = T_3\underset{\sim}{Y}^{(2)}$.

Let A be a $(q-1) \times q$ matrix of rank $q-1$, where q is the number of components of $\underset{\sim}{X}^{(2)}$. Consider the hyperplane $A\underset{\sim}{Y}^{(2)} = \underset{\sim}{0}$. This defines the values of $\underset{\sim}{Y}^{(2)}/Y_{p-q+1}$ where $\underset{\sim}{Y}' = (Y_1, \ldots, Y_p)$. It will be enough to show that for every such matrix A,

$$P[\underset{\sim}{X}^{(1)} \in D_1, \underset{\sim}{X}^{(2)} \in D_2 | A\underset{\sim}{Y}^{(2)} = 0]$$

$$\geq P[\underset{\sim}{X}^{(1)} \in D_1] P[\underset{\sim}{X}^{(2)} \in D_2 | A\underset{\sim}{Y}^{(2)} = \underset{\sim}{0}],$$

that is,

$$P[T_1\underset{\sim}{Y}^{(1)} + T_2\underset{\sim}{Y}^{(2)} \in D_1, T_3\underset{\sim}{Y}^{(2)} \in D_2 | A\underset{\sim}{Y}^{(2)} = \underset{\sim}{0}]$$

$$\geq P[\underset{\sim}{X}^{(1)} \in D_1] P[T_3\underset{\sim}{Y}^{(2)} \in D_2 | A\underset{\sim}{Y}^{(2)} = \underset{\sim}{0}].$$

There exists a convex and symmetric (about the origin) set D_2^* such that

$$T_3\underset{\sim}{Y}^{(2)} \in D_2 \Leftrightarrow \underset{\sim}{Y}^{(2)} \in D_2^*,$$

since T_3 is nonsingular. There exists an orthogonal $q \times q$ matrix L such that

$$\underset{\sim}{Y}^{(2)} = L\underset{\sim}{Y}_*^{(2)}$$

and

$$A\underset{\sim}{Y}^{(2)} = \underset{\sim}{0} \Leftrightarrow Y_p^* = \cdots = Y_{p-q+2}^* = 0$$

where $\underset{\sim}{Y}_*^{(2)'} = (Y_{p-q+1}^*, \ldots, Y_p^*)$. Hence, we shall have to show that (writing $\underset{\sim}{Y}^{(2)}$ for $\underset{\sim}{Y}_*^{(2)}$, without any loss of generality)

$$P[T_1\underset{\sim}{Y}^{(1)} + T_2\begin{pmatrix} Y_{p-q+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in D_1, \begin{pmatrix} Y_{p-q+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in D_2^*]$$

$$\geq P[\underset{\sim}{X}^{(1)} \in D_1] P[\begin{pmatrix} Y_{p-q+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in D_2^*].$$

There exists a positive constant a such that

$$\begin{pmatrix} Y_{p-q+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in D_2^* \Leftrightarrow |Y_{p-q+1}| \leq a \quad (\text{or}, < a).$$

It follows from Khatri's first result that

$$P[T_1\underset{\sim}{Y}^{(1)} + \underset{\sim}{t}_{21}Y_{p-q+1} \in D_1, |Y_{p-q+1}| \leq a]$$
$$\geq P[T_1\underset{\sim}{Y}^{(1)} + \underset{\sim}{t}_{21}Y_{p-q+1} \in D_1]P[|Y_{p-q+1}| \leq a],$$

where $\underset{\sim}{t}_{21}$ is the first column vector of T_2. Note that

$$\mathrm{Cov}(\underset{\sim}{X}^{(1)}) = T_1T_1' + T_2T_2'$$

and

$$\mathrm{Cov}(T_1\underset{\sim}{Y}^{(1)} + \underset{\sim}{t}_{21}Y_{p-q+1}) = T_1T_1' + \underset{\sim}{t}_{21}\underset{\sim}{t}_{21}'.$$

It follows now from Lemma 1 that

$$P[T_1\underset{\sim}{Y}^{(1)} + \underset{\sim}{t}_{21}Y_{p-q+1} \in D_1] \geq P[\underset{\sim}{X}^{(1)} \in D_1].$$

Theorem 2 now follows easily.

It may be mentioned that Khatri's first result can be easily generalized to the case where D_1 and D_2 are hyperspheres with the origin as their centers and without having the assumption that $\mathrm{Cov}(\underset{\sim}{X}^{(1)}, \underset{\sim}{X}^{(2)})$ is of rank 1. The proof is more or less similar to the proof of Khatri's result. It also follows that Theorem 1 holds without having the restriction on $\mathrm{Cov}(\underset{\sim}{X}^{(1)}, \underset{\sim}{X}^{(2)})$.

Trivial generalization of Theorem 2 can be easily proved when Σ is singular.

These results lead us to make the following conjecture.

Let E_1 and E_2 be convex sets in the n-dimensional Euclidean space, symmetric about the origin. For $\underset{\sim}{x}(n \times 1)$, let $f(\underset{\sim}{x}) \geq 0$ be a function such that (i) $f(\underset{\sim}{x}) = f(-\underset{\sim}{x})$, (ii) $\{\underset{\sim}{x} | f(\underset{\sim}{x}) \geq u\} = K_u$ is convex for every u $(0 < u < \infty)$ and (iii) $\int_{E_i} f(\underset{\sim}{x})d\underset{\sim}{x} < \infty$. Then

$$\int_{E_1 \cup E_2} f(\underset{\sim}{x} + k\underset{\sim}{y})d\underset{\sim}{x}$$

decreases as k increases $(0 \leq k \leq 1)$.

This is a generalization of Anderson's Theorem [1].

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